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DIRECT TREATMENT AND DISCRETIZATIONS OF NON-LINEAR SPATIALLY CONTINUOUS SYSTEMS

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Approximate analytical methods for the study of non-linear vibrations of spatially continuous systems with general quadratic and cubic non-linearities are discussed. The cases of an external primary resonance of a non-internally resonant mode and of a sub-harmonically excited two-to-one internal resonance are investigated. It is shown, in a general fashion, that application of the method of multiple scales to the original partial-differential equations and boundary conditions produces the same approximate dynamics as those obtained by applying the reduction method to the full-basis Galerkin-discretized system (using the complete set of eigenfunctions of the associated linear system) or to convenient low-order rectified Galerkin models. As a corollary, it is shown that, due to the effects of the quadratic non-linearities, all of the modes from the relevant eigenspectrum, in principle, contribute to the non-linear motions. Hence, classical low-order Galerkin models may be inadequate to describe quantitatively and qualitatively the dynamics of the original continuous system. Although the direct asymptotic and rectified Galerkin procedures seem to be more "appealing" from a computational standpoint, the full-basis Galerkin discretization procedure furnishes a remarkably interesting spectral representation of the non-linear motions.

1. INTRODUCTION

Elastic systems such as arches, cables, plates, and shells are usually modelled by non-linear partial-differential or integral-partial-differential equations with pertinent boundary conditions. Non-linearities can appear in the governing partial-differential equations, boundary conditions, or both. Non-linearities can be strong or weak. Typically, use of analytical perturbation techniques allows one to construct the approximate dynamics of systems with weak non-linearities or the local dynamics of systems with strong non-linearities.

Within the framework of analytical techniques, non-linear vibrations of continuous (distributed-parameter) systems can be studied either by attacking directly the original partial-differential equations and boundary conditions with

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a reduction method (e.g., the method of multiple scales) or by discretizing the system, first, and, then, by constructing, via a reduction method, approximations of the obtained reduced-order systems. With the first approach (direct treatment), the reduction procedure acts on the temporal dependence of the system without any *a priori* assumption of the form of the solution. With space discretization, the spatial condensation, also referred to as system order reduction, achieved by means of one of the many versions of the method of weighted residuals [1], is a crucial step. It is a common practice to project via the standard or "flat" Galerkin procedure the original infinite-dimensional non-linear system on to a basis forming a complete set of functions usually consisting of the eigenfunctions of the associated linearized system when the boundary conditions are homogeneous. Then, truncation to a finite number of basis functions generates classical low-order models. Inherent limitations of this conventional discretization procedure have been highlighted by a number of works (see references [2–4, 6, 7]).

In the context of buckling problems, Troger and Steindl [2] showed that discretization procedures such as the Rayleigh–Ritz and Galerkin methods can, for some examples, lead to qualitatively incorrect bifurcation diagrams. Hence, these procedures may lead to erroneous conclusions about the structural stability of a system.

Recently, Lacarbonara *et al.* [3], while exploring theoretically and experimentally the response of a fixed–fixed first-mode buckled beam to a primary resonance of its first mode when no internal resonances were activated, showed that direct treatment of the integral-partial-differential equation and associated boundary conditions yielded, for high buckling levels, results in agreement with the experiments whereas some low-order Galerkin-reduced models led to qualitatively erroneous results.

Also in the context of internal resonances, Rega *et al.* [4] showed that a four-mode Galerkin model of multiple internal resonances involving four modes of a suspended elastic cable yielded results in disagreement with the outcomes of a direct treatment of the original system for some classes of motions and their bifurcations.

Nayfeh [5] developed a scheme for constructing reduced-order models of non-linear distributed-parameter systems that overcome the shortcomings of low-order classical Galerkin discretizations. We refer to this method as *rectified Galerkin procedure* for non-linear systems. Application of the method of multiple scales to the constructed rectified models for an Euler–Bernoulli beam resting on a non-linear elastic foundation and a buckled beam yielded results for the modulation equations that agree with those obtained by directly attacking the partial-differential equations and associated boundary conditions. Nayfeh and Lacarbonara [6] summarized the results obtained by low-order classical Galerkin or rectified Galerkin discretizations and direct treatment of six continuous systems.

In studying the non-linear normal modes of a simply supported beam on an elastic foundation with quadratic and cubic non-linearities when no internal resonances were activated, Nayfeh *et al.* [7] showed that approximate results obtained with direct treatment of the governing equations agree with results obtained with a full-basis Galerkin discretization procedure. Later, Pakdemirli and Boyaci [8] attempted an extension of this result to general self-adjoint systems with quadratic and cubic non-linearities when no internal resonances were activated. However, in their analyses, one of the fundamental results was postulated instead of proved.

In this paper, a class of one-dimensional distributed-parameter systems with general quadratic and cubic geometric and quadratic inertia non-linearities and with general homogeneous boundary conditions is considered. The direct treatment, the full-basis Galerkin discretization, and the rectified Galerkin procedures are used to obtain a second-order approximate response of the system to a primary resonance of the *n*th mode when this mode is not involved in internal resonances with any other mode. Also constructed, by using these three procedures, is a first-order approximation of a two-to-one internal resonance involving two modes excited by an external sub-harmonic resonance of order one-half of the high-frequency mode. It is shown, in a general fashion, that the three procedures yield the same results for the approximate dynamics. It is worth noting that these results may be extended to more general non-self-adjoint systems (e.g., moving media) and general bidimensional distributed-parameter systems.

2. A CLASS OF ONE-DIMENSIONAL SYSTEMS

Consider a class of one-dimensional distributed-parameter systems with quadratic and cubic geometric and quadratic inertia non-linearities. Systems with initial curvature, by restricting the analysis to the local dynamics around their initial static equilibrium configurations, belong to this general class. In non-dimensional form, non-linear motions for these systems are governed by

$$\ddot{v} + \mathcal{L}v = \mathcal{N}_2(v, v) + \mathcal{I}_2(\dot{v}, \dot{v}) + \mathcal{N}_3(v, v, v) - c\dot{v} + F(s, t), \tag{1}$$

subject, without loss of generality, to the linear homogeneous boundary conditions

$$\mathcal{B}_1 v = 0 \quad \text{at} \quad s = 0 \quad \text{and} \quad \mathcal{B}_2 v = 0 \quad \text{at} \quad s = 1, \tag{2}$$

where s is the co-ordinate along the centerline of the system (non-dimensionalized with respect to the span); the overdot indicates differentiation with respect to the non-dimensional time t; $v(s, t)$ is the dynamic deflection with respect to the initially straight or curved configuration; the non-dimensional inertia is assumed to be unitary; $\mathcal{L}$ is a linear and homogeneous, self-adjoint and positive-definite differential or integral-differential operator of order 2p; $\mathcal{N}_2$ and $\mathcal{N}_3$ are quadratic and cubic geometric operators, and $\mathcal{I}_2$ is a quadratic inertia operator; $\mathcal{B}_i$ are linear and homogeneous differential boundary operators of order less than or equal to $2\mathrm{p}-1$; c is the linear viscous damping coefficient; and $F(s, t)$ is the forcing function. In general, it is assumed that the non-linear operators do not commute, i.e., $\mathcal{N}_2(v, w) \neq \mathcal{N}_2(w, v)$. Because the linear unforced undamped problem, by virtue of the self-adjoint nature of the linear stiffness operator with given boundary conditions on the appropriate domain with compact support, is self-adjoint, the eigenfunctions $\phi_m(s)$ are mutually orthogonal and have been normalized as follows

$$\int_0^1 \phi_m(s)\phi_n(s)\,\mathrm{d}s = \langle \phi_m \phi_n \rangle = \delta_{mn}, \quad \langle \phi_m\ \mathcal{L}\phi_n \rangle = \omega_n^2 \delta_{mn}, \tag{3}$$

where δ_{mn} is the Kronecker delta. The eigenvalue problem for the frequencies and the mode shapes defines the linear operator $\mathcal{M}$ as

$$\mathcal{M}[\phi;\omega] = (\mathcal{L} - \omega^2 I)\phi, \tag{4}$$

where I is the identity operator.

For a beam resting on a non-linear elastic foundation,

$$\mathcal{L}v = v'''', \quad \mathcal{N}_2(v, v) = -\alpha_2 v^2, \quad \mathcal{N}_3(v, v, v) = -\alpha_3 v^3, \tag{5}$$

where the prime indicates differentiation with respect to the non-dimensional co-ordinate s.

For a shallow arch with the given initial shape $w(s)$, subject to the end-load p, in the pre- or post-buckling condition,

$$\begin{gathered}\mathcal{L}v = v'''' + Pv'' - \psi''\langle v'\psi' \rangle, \quad \mathcal{N}_2(v, v) = v''\langle v'\psi' \rangle + \tfrac{1}{2}\psi''\langle v'^2 \rangle, \\ \mathcal{N}_3(v, v, v) = \tfrac{1}{2} v''\langle v'^2 \rangle,\end{gathered} \tag{6}$$

where

$$P = p + \tfrac{1}{2}\langle w'^2 - \psi'^2 \rangle. \tag{7}$$

The static equilibrium configurations ψ due to the end-load p, in the pre- or post-buckling range, are solutions of a non-linear ordinary-integral-differential equation with the associated boundary conditions.

For a suspended homogeneous elastic cable with small sag-to-span ratios [9], using vector notation,

$$\mathcal{L}\mathbf{v} = -\begin{bmatrix} u_1'' + kb^2\psi''\langle u_1'\psi' \rangle & 0 \\ 0 & u_2'' \end{bmatrix}, \tag{8}$$

$$\mathcal{N}_2(\mathbf{v}, \mathbf{v}) = kb\begin{Bmatrix} u_1''\langle u_1'\psi' \rangle + \tfrac{1}{2}\psi''\langle u_1'^2 + u_2'^2 \rangle \\ u_2''\langle u_1'\psi' \rangle \end{Bmatrix}, \quad \mathcal{N}_3(\mathbf{v}, \mathbf{v}, \mathbf{v}) = \frac{1}{2}k\begin{Bmatrix} u_1''\langle u_1'^2 + u_2'^2 \rangle \\ u_2''\langle u_1'^2 + u_2'^2 \rangle \end{Bmatrix}, \tag{9}$$

$$\mathcal{C}\dot{\mathbf{v}} = \begin{bmatrix} c_1\dot{u}_1 & 0 \\ 0 & c_2\dot{u}_2 \end{bmatrix}, \quad \mathbf{F}(s, t) = \begin{Bmatrix} P_1(s)\cos \Omega t \\ P_2(s)\cos(\Omega t + \tau) \end{Bmatrix}, \tag{10}$$

where $k = EA/H$ (EA is the axial rigidity of the cable and H is the initial static tension), b is the sag-to-span ratio, $\mathbf{v}^{\mathrm{T}} = \{u_1, u_2\}$ denotes the vector of the in-plane (vertical) and out-of plane (horizontal) displacement components and the superscript T indicates the transpose.

3. DIRECT TREATMENT

The method of multiple scales [10, 11] is used to determine a second-order uniform expansion of the solution of equations (1) and (2) when the system is subject to a primary resonance of the nth mode and no active internal resonances engage this mode with any other mode. To this end, the forcing function is assumed to be a pure tone: $F(s,\, t) = F(s)\cos\Omega t$. Moreover, the damping and forcing are scaled as $\varepsilon^2 c$ and $\varepsilon^3 F$, respectively. Note that because resonant terms appear at the third order, the solution does not depend on the non-linear time scale $T_1 = \varepsilon t$. Hence, a solution of equations (1) and (2) is sought in the form

$$v(s, t; \varepsilon) = \varepsilon v_1(s, T_0, T_2) + \varepsilon^2 v_2(s, T_0, T_2) + \varepsilon^3 v_3(s, T_0, T_2) + \cdots, \tag{11}$$

where ε is a small non-dimensional bookkeeping parameter, $T_0 = t$ is a fast scale, and $T_2 = \varepsilon^2 t$ is a slow non-linear time scale. To express the nearness of the primary resonance, the detuning parameter σ is introduced such that $\Omega = \omega_n + \varepsilon^2\sigma$.

Substituting equation (11) into equations (1) and (2), using the independence of the time scales, and equating coefficients of like powers of ε yields:

Order ε:

$$\mathcal{F}(v_1) = D_0^2 v_1 + \mathcal{L}v_1 = 0, \tag{12}$$

Order ε^2:

$$\mathcal{F}(v_2) = \mathcal{N}_2(v_1, v_1) + \mathcal{I}_2(D_0 v_1, D_0 v_1), \tag{13}$$

Order ε^3:

$$\begin{aligned}\mathcal{F}(v_3) = {} & -2D_0 D_2 v_1 - cD_0 v_1 + \mathcal{N}_2(v_1, v_2) + \mathcal{N}_2(v_2, v_1) \\ & + \mathcal{I}_2(D_0 v_1, D_0 v_2) + \mathcal{I}_2(D_0 v_2, D_0 v_1) \\ & + \mathcal{N}_3(v_1, v_1, v_1) + F(s)\cos\Omega T_0,\end{aligned} \tag{14}$$

where $D_n = \partial/\partial T_n$. The boundary conditions at all orders are given by

$$\mathcal{B}_1 v_j = 0 \quad \text{at} \quad s = 0 \quad \text{and} \quad \mathcal{B}_2 v_j = 0 \quad \text{at} \quad s = 1, \quad \text{for} \quad j = 1, 2, \text{ and } 3. \tag{15}$$

Because the nth mode is directly excited and no internal resonances are activable; moreover, because the system is damped and the interest is in the steady-state dynamics for this mode, the generating solution at order ε is assumed to be

$$v_1 = [A_n(T_2)\,\mathrm{e}^{\mathrm{i}\omega_n T_0} + \bar{A}_n(T_2)\,\mathrm{e}^{-\mathrm{i}\omega_n T_0}]\phi_n(s), \tag{16}$$

where the overbar indicates the complex conjugate. Substituting equation (16) into equations (13) and (15) yields the following inhomogeneous problem at second order:

$$\mathcal{F}(v_2) = h_{1n}(s)A_n^2\,\mathrm{e}^{2\mathrm{i}\omega_n T_0} + h_{2n}(s)A_n\bar{A}_n + cc, \tag{17}$$

where cc stands for the complex conjugate of the preceding terms, and

$$h_{1n}(s) = \mathcal{N}_2(\phi_n, \phi_n) - \omega_n^2\mathcal{I}_2(\phi_n, \phi_n), \tag{18}$$

$$h_{2n}(s) = \mathcal{N}_2(\phi_n, \phi_n) + \omega_n^2\mathcal{I}_2(\phi_n, \phi_n), \tag{19}$$

with boundary conditions (15).

The solution of the second-order problem is easily found as

$$v_2 = \Psi_{1n}(s)A_n^2\,\mathrm{e}^{2\mathrm{i}\omega_n T_0} + \Psi_{2n}(s)A_n\bar{A}_n + cc, \tag{20}$$

where the functions Ψ_{jn} are solutions of the following boundary-value differential or integral-differential problems

$$\mathcal{M}[\Psi_{1n}; 2\omega_n] = h_{1n}(s), \qquad \mathcal{M}[\Psi_{2n}; 0] = h_{2n}(s), \tag{21}$$

$$\mathcal{B}_1\Psi_{jn} = 0 \quad \text{at} \quad s = 0 \quad \text{and} \quad \mathcal{B}_2\Psi_{jn} = 0 \quad \text{at} \quad s = 1, \quad \text{for} \quad j = 1 \text{ and } 2. \tag{22}$$

Substituting the second-order solution, equation (20), into the third-order problem, equations (14) and (15), terms that produce resonant effects arise causing the expansion to break down. To render the expansion uniform, a solvability condition is imposed by multiplying the right-hand side of the third-order differential equation by the adjoint $\phi_n(s)\exp(-\mathrm{i}\omega_n T_0)$ and integrating the resulting equation over the normalized space domain [0, 1] (see reference [10]). The result is a complex-valued modulation equation for the amplitude A_n governing the slow dynamics of the system; that is,

$$2\mathrm{i}\omega_n(D_2A_n + \mu A_n) = 8\omega_n\alpha_{nn}A_n^2\bar{A}_n + \tfrac{1}{2}\,f_n\,\mathrm{e}^{\mathrm{i}\sigma T_2}, \tag{23}$$

where $c = 2\mu$, the nth-modal projection of the force is $f_n = \langle\phi_n F\rangle$, and the *effective non-linearity coefficient* is given by

$$\alpha_{nn} = \frac{1}{8\omega_n}(S_{nnn} + 3\Gamma_{nnnn}). \tag{24}$$

In equation (24), the overall softening effects caused by the quadratic non-linearities are expressed as

$$\begin{aligned} S_{nnn} = {} & \langle\phi_n\mathcal{N}_2(\phi_n, \Psi_{1n})\rangle + \langle\phi_n\mathcal{N}_2(\Psi_{1n}, \phi_n)\rangle + 2\langle\phi_n\mathcal{N}_2(\phi_n, \Psi_{2n})\rangle \\ & + 2\langle\phi_n\mathcal{N}_2(\Psi_{2n}, \phi_n)\rangle + 2\omega_n^2\langle\phi_n\mathcal{I}_2(\phi_n, \Psi_{1n})\rangle + 2\omega_n^2\langle\phi_n\mathcal{I}_2(\Psi_{1n}, \phi_n)\rangle. \end{aligned} \tag{25}$$

The coefficient Γ_{nnnn} can be obtained by putting $k = l = m = n$ in

$$\Gamma_{klmn} = \Gamma_{nklm} = \langle\phi_n\mathcal{N}_3(\phi_k, \phi_l, \phi_m)\rangle. \tag{26}$$

Then, using the polar form $A_n = (1/2)a_n\exp[\mathrm{i}(\sigma T_2 - \gamma_n)]$ gives the real-valued modulation equations for the amplitude and phase

$$D_2 a_n = -\mu a_n + \frac{1}{2}\frac{f_n}{\omega_n}\sin\gamma_n, \tag{27}$$

$$a_n(D_2\gamma_n) = a_n\sigma + \alpha_{nn}a_n^3 + \frac{1}{2}\frac{f_n}{\omega_n}\cos\gamma_n. \tag{28}$$

Therefore, the steady-state responses (i.e., the fixed points of equations (27) and (28) with $D_2 a_n = 0$ and $a_n(D_2\gamma_n) = 0$) are solutions of the following frequency-response equation:

$$\sigma = -\alpha_{nn}a_n^2 \pm \left(\frac{f_n^2}{4\omega_n^2 a_n^2} - \mu^2\right)^{1/2}. \tag{29}$$

Combining equations (11), (16), and (20), using the polar form for A_n, and the resonance detuning condition, one can express the displacement field, to second order, as

$$v(s,t) = \varepsilon a_n \cos(\Omega t - \gamma_n)\phi_n(s) + \tfrac{1}{2}\varepsilon^2 a_n^2[\cos 2(\Omega t - \gamma_n)\Psi_{1n}(s) + \Psi_{2n}(s)] + \cdots \tag{30}$$

Note that the effects of the quadratic non-linearities are twofold: (i) they produce in the displacement field (30) a drift term—$\propto a_n^2\Psi_{2n}(s)$—and an overtone term—$\propto a_n^2\cos 2(\Omega t - \gamma_n)\Psi_{1n}(s)$—while retaining, as will be shown in the later analyses, the full spectrum of the eigenmodes of the system in the functions Ψ_{1n} and Ψ_{2n}; (ii) they are responsible for contributions from all of the modes of the system, through Ψ_{1n} and Ψ_{2n}, to the softening effects, expressed by equation (25), in the effective non-linearity coefficient (24).

Next, a first-order approximate solution of the system is constructed when a two-to-one internal resonance is excited by a sub-harmonic resonance of order one-half of the high-frequency mode. To quantify the nearness of the resonances, the detuning parameters σ_1 and σ_2 are introduced such that

$$\omega_n = 2\omega_m + \varepsilon\sigma_1 \quad \text{and} \quad \Omega = 2\omega_n + \varepsilon\sigma_2. \tag{31}$$

For these resonances, proper scaling of the forcing and damping requires that the former appear at first order and the latter at second order.

The first-order solution is expressed as

$$v_1(s,t) = A_m(T_1)\phi_m(s)\,e^{i\omega_m T_0} + A_n(T_1)\phi_n(s)e^{i\omega_n T_0} + \Psi_p(s)e^{i\Omega T_0} + cc, \tag{32}$$

where $\Psi_p(s)$ is the solution of the boundary-value problem

$$\mathcal{M}[\Psi_p;\Omega] = \tfrac{1}{2}F(s), \tag{33}$$

with boundary conditions (22).

Substituting equation (32) into equation (13) (where the damping term was added) and accounting for the resonance detunings (31), elimination of the terms that produce secular effects at second order leads to the following two coupled solvability conditions:

$$2\mathrm{i}\omega_m(D_1A_m + \mu A_m) = S_1 A_n \bar{A}_m \mathrm{e}^{\mathrm{i}\sigma_1 T_1}, \tag{34}$$

$$2\mathrm{i}\omega_n(D_1A_n + \mu A_n) = S_2 A_m^2 \mathrm{e}^{-\mathrm{i}\sigma_1 T_1} + S_3 \bar{A}_n \mathrm{e}^{\mathrm{i}\sigma_2 T_1}, \tag{35}$$

where the non-linear interaction coefficients are given by

$$S_1 = \Lambda^+_{mmn} + \Lambda^+_{mnm}, \quad S_2 = \Lambda^-_{nmm}, \tag{36}$$

$$S_3 = \langle \phi_n \mathcal{N}_2(\phi_n, \Psi_p)\rangle + \langle \phi_n \mathcal{N}_2(\Psi_p, \phi_n)\rangle + \Omega\omega_n \langle \phi_n \mathcal{I}_2(\phi_n, \Psi_p)\rangle + \Omega\omega_n \langle \phi_n \mathcal{I}_2(\Psi_p, \phi_n)\rangle, \tag{37}$$

with

$$\Lambda^{\pm}_{klm} = \Lambda_{klm} \pm \omega_l\omega_m \Delta_{klm}, \tag{38}$$

$$\Lambda_{klm} = \langle \phi_k \mathcal{N}_2(\phi_l, \phi_m)\rangle, \quad \text{and} \quad \Delta_{klm} = \langle \phi_k \mathcal{I}_2(\phi_l, \phi_m)\rangle. \tag{39}$$

Using the polar transformations

$$A_m = \tfrac{1}{2} a_m \mathrm{e}^{1/4\mathrm{i}(2\sigma_1 T_1 + \sigma_2 T_1 - 2\gamma_1 - \gamma_2)} \quad \text{and} \quad A_n = \tfrac{1}{2} a_n \mathrm{e}^{1/2\mathrm{i}(\sigma_2 T_1 - \gamma_2)}, \tag{40}$$

one can express the displacement field, to first order, as

$$v(s,t) = \varepsilon a_m \cos\tfrac{1}{4}(\Omega t - \gamma_2 - 2\gamma_1)\phi_m(s) + \varepsilon a_n \cos\tfrac{1}{2}(\Omega t - \gamma_2)\phi_n(s) + 2\varepsilon \Psi_p(s)\cos\Omega t + \cdots \tag{41}$$

where a_m, a_n, γ_1, and γ_2 are solutions of the modulation equations (34) and (35).

In the next sections, it is shown that the function Ψ_p embodies modal contributions, in principle, from the full eigenspectrum. Therefore, the one-half external sub-harmonic resonance acts to capture, to first order, contributions from all of the modes to (i) the displacement field (41) and (ii) to the *effective sub-harmonic resonance excitation amplitude* S_3, given by equation (37).

4. FULL-BASIS GALERKIN DISCRETIZATION

Taking the set of the eigenfunctions of the associated linear problem as a complete set for discretizing the system via the Galerkin method, it is postulated that the solution can be represented as

$$v(s,t) = \sum_{k=1}^{\infty} \xi_k(t)\phi_k(s). \tag{42}$$

Hence, using the Galerkin method and employing the ortho-normality of the eigenfunctions, one obtains an infinite set of non-linearly coupled ordinary-differential equations for the generalized co-ordinates $\xi_k(t)$

$$\ddot{\xi}_k + \omega_k^2 \xi_k = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} (\Lambda_{klm} \xi_l \xi_m + \Delta_{klm} \dot{\xi}_l \dot{\xi}_m) + \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Gamma_{klmn} \xi_l \xi_m \xi_n - c\dot{\xi}_k + f_k \cos \Omega t, \qquad k = 1, 2, \ldots, \infty, \quad (43)$$

where Λ_{klm} and Δ_{klm} are given by equations (39), Γ_{klmn} is given by equation (26), and $f_k = \langle \phi_k F \rangle$.

Using the method of multiple scales to construct a second-order approximation of the response of system (43) to a primary resonance of the nth mode, one obtains

$$\hat{v}(s, t) = \varepsilon \hat{a}_n \cos(\Omega t - \hat{\gamma}_n) \phi_n(s) + \tfrac{1}{2} \varepsilon^2 \hat{a}_n^2 [\cos 2(\Omega t - \hat{\gamma}_n) \hat{\Psi}_{1n}(s) + \hat{\Psi}_{2n}(s)] + \cdots \quad (44)$$

where

$$\hat{\Psi}_{1n}(s) = \sum_{k=1}^{\infty} \frac{\Lambda_{knn}^-}{\omega_k^2 - 4\omega_n^2} \phi_k(s) \quad \text{and} \quad \hat{\Psi}_{2n}(s) = \sum_{k=1}^{\infty} \frac{\Lambda_{knn}^+}{\omega_k^2} \phi_k(s). \quad (45)$$

The modulation equations for the amplitude and phase of the motion, expressed by equations (44) and (45), are the same as equations (27) and (28) obtained with the direct treatment once the softening term S_{nnn} is replaced in the effective non-linearity coefficient (24) with

$$\hat{S}_{nnn} = \sum_{j=1}^{\infty} \left[(\Lambda_{nnj} + \Lambda_{njn}) \left(\frac{2\Lambda_{jnn}^+}{\omega_j^2} + \frac{\Lambda_{jnn}^-}{\omega_j^2 - 4\omega_n^2} \right) + 2\omega_n^2 (\Delta_{nnj} + \Delta_{njn}) \frac{\Lambda_{jnn}^-}{\omega_j^2 - 4\omega_n^2} \right], \quad (46)$$

and $\Lambda_{klm}^{\pm}$ is given by equation (38).

It is worth noting that a finite-dimensional discretization procedure would yield finite-order models which can be easily extracted from the infinite-dimensional solution, equations (44)–(46), by retaining a finite number of modes.

The effective non-linearity coefficient (24) consists of two terms: one term, $3\Gamma_{nnnn}/8\omega_n$, generated by the cubic non-linearity, and one term, $S_{nnn}/8\omega_n$ if calculated with the direct procedure or $\hat{S}_{nnn}/8\omega_n$ if calculated with the full-basis discretization, caused by the quadratic non-linearities. The first term, which is the same, to second order, if calculated with the full-basis Galerkin discretization or the direct procedure (or with a low-order Galerkin discretization), can be shown to be negative; therefore, it produces a hardening behaviour. On the other hand, the softening term depends evidently on the order (i.e., number of modes) of the discretization procedure. Because the effective non-linearity coefficient depends on the relative magnitudes of the hardening (cubic) and softening (quadratic) terms, there is a possibility of a sign difference in this coefficient depending on the order of the discretization procedure employed. Consequently, *full-basis and low-order Galerkin discretization procedures, depending on the order* *of the latter, may yield qualitatively different dynamics for primary resonances of a non-internally resonant mode.*

Next, constructing a first-order uniform expansion of equations (43), when a two-to-one internal resonance is excited by a one-half sub-harmonic resonance of the high-frequency mode, gives

$$\hat{v}(s,t) = \varepsilon \hat{a}_m \cos\tfrac{1}{4}(\Omega t - \hat{\gamma}_2 - 2\hat{\gamma}_1)\phi_m(s) + \varepsilon \hat{a}_n \cos\tfrac{1}{2}(\Omega t - \hat{\gamma}_2)\phi_n(s) + 2\varepsilon \hat{\Psi}_p(s)\cos\Omega t + \cdots, \tag{47}$$

where

$$\hat{\Psi}_p(s) = \sum_{k=1}^{\infty} G_k \phi_k(s) \tag{48}$$

and

$$G_k = \frac{1}{2}\frac{f_k}{\omega_k^2 - \Omega^2}. \tag{49}$$

The complex-valued modulation equations for the amplitudes $\hat{A}_m$ and $\hat{A}_n$ of the interacting modes are the same as equations (34) and (35) once S_3 is replaced with

$$\hat{S}_3 = \sum_{k=1}^{\infty}[\Lambda_{nnk} + \Lambda_{nkn} + \Omega\omega_n(\Delta_{nnk} + \Delta_{nkn})]G_k. \tag{50}$$

It is clear that due to the external one-half sub-harmonic resonance: (i) the effects of all modes are embodied in the displacement field (47) through the function $\hat{\Psi}_p$, given by equation (48), and (ii) the effective sub-harmonic resonance excitation amplitude (50) also captures contributions from all of the eigenmodes. In a low-order Galerkin discretization, the function $\hat{\Psi}_p$ and the effective sub-harmonic resonance excitation amplitude $\hat{S}_3$ would be expressed by finite summations with the number of terms being equal to the number of retained modes or discretizing functions. On the other hand, the non-linear interaction coefficients $\hat{S}_1$ and $\hat{S}_2$, by inspection of equation (36), do not depend on the order of the discretization procedure. Quantitative discrepancies in the computation of $\hat{S}_3$ may affect the eigenvalue structure of the modulation equations (34) and (35) for the non-linear modal coupling. Therefore, *full-basis and low-order Galerkin discretizations may produce, even to first order, qualitatively different dynamics for the sub-harmonically excited two-to-one internal resonance.*

5. LOW-ORDER RECTIFIED GALERKIN PROCEDURE

Using the rectified Galerkin method proposed by Nayfeh [5], one obtains the following single-mode model for the primary resonance of the nth mode:

$$\ddot{\eta}_n + \omega_n^2\eta_n = \Lambda_{nnn}\eta_n^2 + \Delta_{nnn}\dot{\eta}_n^2 + \tilde{\Gamma}_{nnnn}\eta_n^3 + \Pi_{nnnn}\eta_n\dot{\eta}_n^2 - c\dot{\eta}_n + f_n\cos\Omega t, \tag{51}$$

where higher-order terms for a second-order approximation of the primary resonance were neglected and $\tilde{\Gamma}_{nnnn}$ and Π_{nnnn} are given by

$$\tilde{\Gamma}_{nnnn} = \Gamma_{nnnn} + \langle \phi_n \mathcal{N}_2(\phi_n, \psi_{nn}) \rangle + \langle \phi_n \mathcal{N}_2(\psi_{nn}, \phi_n) \rangle \tag{52}$$

and

$$\begin{aligned}\Pi_{nnnn} = {} & \langle \phi_n \mathcal{N}_2(\phi_{nn}, \chi_{nn}) \rangle + \langle \phi_n \mathcal{N}_2(\chi_{nn}, \phi_n) \rangle \\ & + 2\langle \phi_n \mathcal{I}_2(\phi_n, \psi_{nn} - \omega_n^2 \chi_{nn}) \rangle + 2\langle \phi_n \mathcal{I}_2(\psi_{nn} - \omega_n^2 \chi_{nn}, \phi_n) \rangle.\end{aligned} \tag{53}$$

The displacement field, to second order, is expressed as

$$\tilde{v}(s,t) = \eta_n(t)\phi_n(s) + [\eta_n(t)^2 \psi_{nn}(s) + \dot{\eta}_n(t)^2 \chi_{nn}(s)] + \cdots, \tag{54}$$

where the functions ψ_{nn} and χ_{nn} are solutions of the following coupled two-point boundary-value problems [5]:

$$\mathcal{L}\psi_{nn} - 2\omega_n^2 \psi_{nn} + 2\omega_n^4 \chi_{nn} = \mathcal{N}_2(\phi_n, \phi_n) - \Lambda_{nnn}\phi_n, \tag{55}$$

$$\mathcal{L}\chi_{nn} + 2\psi_{nn} - 2\omega_n^2 \chi_{nn} = \mathcal{I}_2(\phi_n, \phi_n) - \Delta_{nnn}\phi_n, \tag{56}$$

with boundary conditions for ψ_{nn} and χ_{nn} given by equations (22). The differential equations (55) and (56) can be decoupled by introducing the linear transformations [5]

$$\psi_{nn} = \frac{1}{4}(h_{nn} + 2g_{nn}) \quad \text{and} \quad \chi_{nn} = \frac{h_{nn} - 2g_{nn}}{4\omega_n^2}. \tag{57}$$

The resulting decoupled system is

$$\mathcal{M}[g_{nn}; 2\omega_n] = \mathcal{N}_2(\phi_n, \phi_n) - \omega_n^2 \mathcal{I}_2(\phi_n, \phi_n) - \Lambda^-_{nnn}\phi_n, \tag{58}$$

$$\mathcal{M}[h_{nn}; 0] = 2\mathcal{N}_2(\phi_n, \phi_n) + 2\omega_n^2 \mathcal{I}_2(\phi_n, \phi_n) - 2\Lambda^+_{nnn}\phi_n, \tag{59}$$

with boundary conditions for g_{nn} and h_{nn} given by equations (22). Note that in equation (51) the rectification of the Galerkin method has produced an additional cubic geometric term—$\propto(\tilde{\Gamma}_{nnnn} - \Gamma_{nnnn})$—and a new cubic inertia term—$\propto \Pi_{nnnn}$.

Applying the method of multiple scales to equation (51) to produce a second-order approximate solution gives

$$\begin{aligned}\eta_n = {} & \varepsilon\eta_{n1} + \varepsilon^2 \eta_{n2} + \cdots = \varepsilon\tilde{a}_n \cos(\Omega t - \tilde{\gamma}_n) \\ & + \varepsilon^2 \frac{1}{6\omega_n^2} \tilde{a}_n^2 [3\Lambda^+_{nnn} - \Lambda^-_{nnn} \cos 2(\Omega t - \tilde{\gamma}_n)] + \cdots\end{aligned} \tag{60}$$

where the amplitude $\tilde{a}_n$ and the phase $\tilde{\gamma}_n$ are fixed points of the same modulation equations as equations (27) and (28) obtained with the direct treatment once one replaces the softening term S_{nnn} in the effective non-linearity coefficient (24) with

$$\tilde{S}_{nnn} = \frac{2}{3\omega_n^2}\Lambda_{nnn}(6\Lambda_{nnn}^{+} - \Lambda_{nnn}^{-}) - \frac{4}{3}\Lambda_{nnn}^{-}\Delta_{nnn} + 3\tilde{\Gamma}_{nnnn} - 3\Gamma_{nnnn} + \omega_n^2\Pi_{nnnn}. \quad (61)$$

Hence, making use of some trigonometric identities and of equation (57), one obtains for the displacement field

$$\tilde{v}(s,t) = \varepsilon\tilde{a}_n\cos(\Omega t - \tilde{\gamma}_n)\phi_n(s) + \tfrac{1}{2}\varepsilon^2\tilde{a}_n^2[\cos 2(\Omega t - \tilde{\gamma}_n)\tilde{\Psi}_{1n}(s) + \tilde{\Psi}_{2n}(s)] + \cdots, \quad (62)$$

where

$$\tilde{\Psi}_{1n} = g_{nn} - \frac{\Lambda_{nnn}^{-}}{3\omega_n^2}\phi_n \quad \text{and} \quad \tilde{\Psi}_{2n} = \frac{1}{2}h_{nn} + \frac{\Lambda_{nnn}^{+}}{\omega_n^2}\phi_n. \quad (63)$$

Next, a two-mode rectified Galerkin model is used to construct a first-order uniform expansion of the two-to-one internal resonance excited by a subharmonic resonance of order one-half of the high-frequency mode.

The displacement field, to first order, is given by

$$\tilde{v}(s,t) = \eta_m(t)\phi_m(s) + \eta_n(t)\phi_n(s) + \zeta(s)\cos\Omega t + \cdots, \quad (64)$$

where $\zeta(s)$ is the solution of the boundary-value problem

$$\mathcal{M}[\zeta;\Omega] = F(s) - f_m\phi_m(s) - f_n\phi_n(s), \quad (65)$$

with boundary conditions for ζ given by equations (22). The generalized coordinates η_r in equation (64) are solutions of the following two-mode rectified Galerkin model:

$$\ddot{\eta}_r + \omega_r^2\eta_r = \sum_{i,j}[\Lambda_{rij}\eta_i\eta_j + \Delta_{rij}\dot{\eta}_i\dot{\eta}_j] - c\dot{\eta}_r + \left[f_r + \sum_j p_{rj}\eta_j\right]\cos\Omega t - \sum_j q_{rj}\dot{\eta}_j\sin\Omega t,$$
$$\text{for} \quad r = m \quad \text{and} \quad n, \quad (66)$$

where higher-order terms for a first-order approximation of the subharmonically excited two-to-one internal resonance were neglected and the summations are extended to the mth and nth modes involved in the internal resonance. Furthermore,

$$p_{rj} = \langle\phi_r\mathcal{N}_2(\phi_j,\zeta)\rangle + \langle\phi_r\mathcal{N}_2(\zeta,\phi_j)\rangle \quad \text{and} \quad q_{rj} = \Omega\langle\phi_r\mathcal{I}_2(\phi_j,\zeta)\rangle + \Omega\langle\phi_r\mathcal{I}_2(\zeta,\phi_j)\rangle. \quad (67)$$

It follows from equations (66) that the Galerkin rectification has generated in each modal equation two additional multiplicative-type excitation components; namely, $(p_{rm}\eta_m + p_{rn}\eta_n)\cos\Omega t$ and $-(q_{rm}\dot{\eta}_m + q_{rn}\dot{\eta}_n)\sin\Omega t$.

Applying the method of multiple scales to equations (66) to construct a first-order approximation of the two-to-one internal resonance gives

$$\eta_r = \varepsilon\tilde{A}_r e^{i\omega_r T_0}\phi_r(s) + \varepsilon G_r e^{i\Omega T_0} + \cdots, \quad \text{for} \quad r = m \quad \text{and} \quad n, \quad (68)$$

where G_r is given by equation (49). The complex-valued modulation equations for the amplitudes $\tilde{A}_m$ and $\tilde{A}_n$ of the interacting modes are the same as equations (34) and (35) once one replaces S_3 with

$$\tilde{S}_3 = 2G_n(\Lambda_{nnn} + \omega_n \Omega \Delta_{nnn}) + G_m(\Lambda_{nnm} + \Lambda_{nmn}) + \omega_n \Omega G_m(\Delta_{nnm} + \Delta_{nmn}) + \tfrac{1}{2} p_{nn} + \tfrac{1}{2}\omega_n q_{nn}. \tag{69}$$

Using the polar transformations (40) and equation (64), one can express the displacement field as

$$\tilde{v}(s,t) = \varepsilon \tilde{a}_m \cos \tfrac{1}{4}(\Omega t - \tilde{\gamma}_2 - 2\tilde{\gamma}_1)\phi_m(s) + \varepsilon \tilde{a}_n \cos \tfrac{1}{2}(\Omega t - \tilde{\gamma}_2)\phi_n(s) + 2\varepsilon \tilde{\Psi}_p(s) \cos \Omega t + \cdots, \tag{70}$$

where

$$\tilde{\Psi}_p(s) = \tfrac{1}{2}\zeta(s) + G_m \phi_m(s) + G_n \phi_n(s). \tag{71}$$

6. EQUIVALENCE BETWEEN DIRECT TREATMENT, FULL-BASIS, AND LOW-ORDER RECTIFIED GALERKIN DISCRETIZATIONS

In the next two sections, it is shown that the approximate solutions obtained with the direct treatment, the full-basis Galerkin discretization, and the rectified Galerkin procedure are equivalent either in the case of no internal resonances or in the case of a sub-harmonically excited two-to-one internal resonance. To show that the approximate displacement fields obtained with the three methods are equivalent, it is shown that the complex-valued amplitude of the only excited mode or the amplitudes of the interacting modes are the same regardless of the method employed and the higher-order spatial corrections due to the non-linearities are the same. It has already been shown that, for both resonances, the modulation equations governing the slow dynamics of the complex-valued amplitudes are formally equivalent for each of the method employed. Hence, one needs to show that the softening term in the effective non-linearity coefficient, in the case of no internal resonances, or the effective sub-harmonic resonance excitation amplitude, in the case of a two-to-one internal resonance, are the same regardless of the procedure used. In addition, one needs to show that the higher-order spatial corrections obtained with the different procedures are the same.

6.1. THE CASE OF NO INTERNAL RESONANCES

Part I. *It is shown that*

$$\hat{\Psi}_{1n}(s) = \Psi_{1n}(s) = \tilde{\Psi}_{1n}(s) \tag{72}$$

and

$$\hat{\Psi}_{2n}(s) = \Psi_{2n}(s) = \tilde{\Psi}_{2n}(s). \tag{73}$$

To prove equations (72) and (73), it is shown that $\hat{\Psi}_{1n}$ and $\hat{\Psi}_{2n}$, $\tilde{\Psi}_{1n}$ and $\tilde{\Psi}_{2n}$ are solutions of the differential boundary-value problems given by equations (21) with boundary conditions (22). That is,

$$\mathcal{M}[\hat{\Psi}_{1n}(s); 2\omega_n] = h_{1n}(s), \qquad \mathcal{M}[\hat{\Psi}_{2n}(s); 0] = h_{2n}(s), \tag{74}$$

$$\mathcal{M}[\tilde{\Psi}_{1n}(s); 2\omega_n] = h_{1n}(s), \qquad \mathcal{M}[\tilde{\Psi}_{2n}(s); 0] = h_{2n}(s), \tag{75}$$

with boundary conditions for $\hat{\Psi}_{jn}$ and $\tilde{\Psi}_{jn}$, for $j = 1$, 2, given by equations (22).

Note that $\hat{\Psi}_{1n}$ and $\hat{\Psi}_{2n}$ satisfy identically the boundary conditions. Applying the operator $\mathcal{M}[\Psi;\, 2\omega_n]$ to $\hat{\Psi}_{1n}$ gives

$$\mathcal{M}[\hat{\Psi}_{1n}; 2\omega_n] = \sum_{k=1}^{\infty} \frac{\Lambda^-_{knn}}{\omega_k^2 - 4\omega_n^2} \mathcal{M}[\phi_k; 2\omega_n] = \sum_{k=1}^{\infty} \frac{\Lambda^-_{knn}}{\omega_k^2 - 4\omega_n^2} (\omega_k^2 - 4\omega_n^2)\phi_k$$
$$= \sum_{k=1}^{\infty} \Lambda^-_{knn} \phi_k(s), \tag{76}$$

where use of the fact that $\mathcal{M}[\phi_k;\, 2\omega_n] = (\omega_k^2 - 4\omega_n^2)\phi_k$ was made. Expressing $h_{1n}(s)$, given by equation (18), in the basis of the eigenfunctions ϕ_k gives

$$h_{1n}(s) = \sum_{k=1}^{\infty} [\langle \phi_k \mathcal{N}_2(\phi_n, \phi_n) \rangle - \omega_n^2 \langle \phi_k \mathcal{I}_2(\phi_n, \phi_n) \rangle] \phi_k = \sum_{k=1}^{\infty} \Lambda^-_{knn} \phi_k(s). \tag{77}$$

Therefore, according to equations (76) and (77), it is concluded that the first equality of equation (72) is true.

Similarly, noting that, according to equation (19), $h_{2n}(s) = \sum_{k=1}^{\infty} \Lambda^+_{knn} \phi_k$, by applying the operator $\mathcal{M}[\Psi;\, 0]$ to $\hat{\Psi}_{2n}$ given by equation (45), one obtains

$$\mathcal{M}[\hat{\Psi}_{2n}; 0] = \sum_{k=1}^{\infty} \frac{\Lambda^+_{knn}}{\omega_k^2} \mathcal{M}[\phi_k; 0] = \sum_{k=1}^{\infty} \Lambda^+_{knn} \phi_k = h_{2n}(s). \tag{78}$$

Consequently, *the functions* $\hat{\Psi}_{1n}(s)$ *and* $\hat{\Psi}_{2n}(s)$ *are spectral realizations, in the basis of the eigenfunctions* ϕ_k, *of* $\Psi_{1n}(s)$ *and* $\Psi_{2n}(s)$.

Nayfeh *et al*. [7] used a Fourier-series expansion of Ψ_{1n} and Ψ_{2n} to show the identities $\Psi_{jn} = \hat{\Psi}_{jn}$ in the case of unforced undamped vibrations of a beam resting on a non-linear elastic foundation with quadratic and cubic non-linearities.

Next, substituting equations (63) into equations (75), after some manipulations, the resulting equations become

$$\mathcal{M}[g_{nn}; 2\omega_n] = \mathcal{N}_2(\phi_n, \phi_n) - \omega_n^2 \mathcal{I}_2(\phi_n, \phi_n) - \Lambda^-_{nnn} \phi_n, \tag{79}$$

$$\mathcal{M}[h_{nn}; 0] = 2\mathcal{N}_2(\phi_n, \phi_n) + 2\omega_n^2 \mathcal{I}_2(\phi_n, \phi_n) - 2\Lambda^+_{nnn} \phi_n, \tag{80}$$

with boundary conditions for g_{nn} and h_{nn} given by equations (22). Equations (79) and (80) with boundary conditions (22) are identically satisfied in force of equations (58), (59) and (22).

Part II. *It is shown that*

$$\hat{S}_{nnn} = S_{nnn} = \tilde{S}_{nnn}. \tag{81}$$

Substituting the identities (72) and (73) and equations (45) into S_{nnn} given by equation (25) yields

$$\begin{aligned} S_{nnn} = \sum_{k=1}^{\infty} \Bigg\{ & \frac{\Lambda_{knn}^{-}}{\omega_k^2 - 4\omega_n^2} [\langle \phi_n \mathcal{N}_2(\phi_n, \phi_k) \rangle + \langle \phi_n \mathcal{N}_2(\phi_k, \phi_n) \rangle \\ & + 2\omega_n^2 (\langle \phi_n \mathcal{I}_2(\phi_n, \phi_k) \rangle + \langle \phi_n \mathcal{I}_2(\phi_k, \phi_n) \rangle)] \\ & + 2 \frac{\Lambda_{knn}^{+}}{\omega_k^2} [\langle \phi_n \mathcal{N}_2(\phi_n, \phi_k) \rangle + \langle \phi_n \mathcal{N}_2(\phi_k, \phi_n) \rangle] \Bigg\}. \end{aligned} \tag{82}$$

Using the definitions (39) for Λ_{klm} and Δ_{klm}, it is verified, with some further algebra, that, according to equations (46) and (82), the first equality of equation (81) is true.

Next, it is shown that $\tilde{S}_{nnn} = S_{nnn}$. Considering the definitions (52) and (53), equation (61) for the softening term of the single-mode rectified model becomes

$$\begin{aligned} \tilde{S}_{nnn} = & \frac{2}{3\omega_n^2} \Lambda_{nnn} (6\Lambda_{nnn}^{+} - \Lambda_{nnn}^{-}) - \tfrac{4}{3} \Lambda_{nnn}^{-} \Delta_{nnn} \\ & + 3 \langle \phi_n \mathcal{N}_2(\phi_n, \psi_{nn}) \rangle + 3 \langle \phi_n \mathcal{N}_2(\psi_{nn}, \phi_n) \rangle \\ & + \omega_n^2 \langle \phi_n \mathcal{N}_2(\phi_n, \chi_{nn}) \rangle + \omega_n^2 \langle \phi_n \mathcal{N}_2(\chi_{nn}, \phi_n) \rangle \\ & + 2\omega_n^2 \langle \phi_n \mathcal{I}_2(\phi_n, \psi_{nn} - \omega_n^2 \chi_{nn}) \rangle + 2\omega_n^2 \langle \phi_n \mathcal{I}_2(\psi_{nn} - \omega_n^2 \chi_{nn}, \phi_n) \rangle. \end{aligned} \tag{83}$$

Using equations (57) and (63), one obtains the following intermediate results

$$\psi_{nn} = \frac{1}{2} (\Psi_{1n} + \Psi_{2n}) + \frac{1}{2\omega_n^2} \left(\frac{\Lambda_{nnn}^{-}}{3} - \Lambda_{nnn}^{+} \right) \phi_n, \tag{84}$$

$$\chi_{nn} = \frac{1}{2\omega_n^2} (\Psi_{2n} - \Psi_{1n}) - \frac{1}{2\omega_n^4} \left(\frac{\Lambda_{nnn}^{-}}{3} + \Lambda_{nnn}^{+} \right) \phi_n, \tag{85}$$

and

$$\psi_{nn} - \omega_n^2 \chi_{nn} = \Psi_{1n} + \frac{1}{3\omega_n^2} \Lambda_{nnn}^{-} \phi_n. \tag{86}$$

Substituting equations (84)–(86) into equation (83) yields

$$\begin{aligned} \tilde{S}_{nnn} = & \langle \phi_n \mathcal{N}_2(\phi_n, \Psi_{1n}) \rangle + \langle \phi_n \mathcal{N}_2(\Psi_{1n}, \phi_n) \rangle \\ & + 2 \langle \phi_n \mathcal{N}_2(\phi_n, \Psi_{2n}) \rangle + 2 \langle \phi_n \mathcal{N}_2(\Psi_{2n}, \phi_n) \rangle \\ & + 2\omega_n^2 \langle \phi_n \mathcal{I}_2(\phi_n, \Psi_{1n}) \rangle + 2\omega_n^2 \langle \phi_n \mathcal{I}_2(\Psi_{1n}, \phi_n) \rangle = S_{nnn}, \end{aligned} \tag{87}$$

where use of equation (25) was made.

6.2. THE CASE OF INTERNAL RESONANCES

Part I. *It is shown that*

$$\hat{\Psi}_p = \Psi_p = \tilde{\Psi}_p. \tag{88}$$

Applying the operator $\mathcal{M}[\Psi;\Omega]$ to $\hat{\Psi}_p$, expressed by equation (48), one obtains

$$\mathcal{M}[\hat{\Psi}_p;\Omega] = \frac{1}{2}\sum_{k=1}^{\infty}\frac{f_k}{\omega_k^2-\Omega^2}\mathcal{M}[\phi_k;\Omega]. \tag{89}$$

Because $\mathcal{M}[\phi_k;\Omega] = (\omega_k^2-\Omega^2)\phi_k$, then, equation (89) can be rewritten as

$$\mathcal{M}[\hat{\Psi}_p;\Omega] = \frac{1}{2}\sum_{k=1}^{\infty}\frac{f_k}{\omega_k^2-\Omega^2}(\omega_k^2-\Omega^2)\phi_k(s) = \sum_{k=1}^{\infty}\frac{1}{2}f_k\phi_k(s) = \frac{1}{2}F(s). \tag{90}$$

Similarly, applying the operator $\mathcal{M}[\Psi;\Omega]$ to $\tilde{\Psi}_p$ expressed by equation (71), one obtains, after some manipulations,

$$\mathcal{M}[\zeta;\Omega] = F(s) - f_m\phi_m(s) - f_n\phi_n(s). \tag{91}$$

This equation, in force of equation (65), is identically satisfied.

Part II. *It is shown that*

$$\hat{S}_3 = S_3 = \tilde{S}_3. \tag{92}$$

Using the identity $\Psi_p = \hat{\Psi}_p$, expressed by equation (88), and substituting the spectral counterpart of Ψ_p, equation (48), into equation (37), the effective subharmonic resonance excitation amplitude becomes

$$S_3 = \frac{1}{2}\sum_{k=1}^{\infty}\frac{f_k}{\omega_k^2-\Omega^2}[\langle\phi_n\mathcal{N}_2(\phi_n,\phi_k)\rangle + \langle\phi_n\mathcal{N}_2(\phi_k,\phi_n)\rangle$$

$$+\,\Omega\omega_n\langle\phi_n\mathcal{I}_2(\phi_n,\phi_k)\rangle + \Omega\omega_n\langle\phi_n\mathcal{I}_2(\phi_k,\phi_n)\rangle]. \tag{93}$$

Using the definitions (39) for Λ_{klm} and Δ_{klm}, one can rewrite equation (93) as

$$S_3 = \sum_{k=1}^{\infty}[\Lambda_{nkn} + \Lambda_{nnk} + \Omega\omega_n(\Delta_{nkn} + \Delta_{nnk})]G_k = \hat{S}_3, \tag{94}$$

where equation (50) was used.

Next, it is shown that $\tilde{S}_3 = S_3$. To this end, from equation (71),

$$\zeta = 2\tilde{\Psi}_p - 2G_m\phi_m - 2G_n\phi_n. \tag{95}$$

Then, using the definitions (67) and the identity $\tilde{\Psi}_p = \Psi_p$, expressed by equation (88), appropriate manipulations of equation (69) lead to

$$\tilde{S}_3 = \langle \phi_n \mathcal{N}_2(\phi_n, \Psi_p) \rangle + \langle \phi_n \mathcal{N}_2(\Psi_p, \phi_n) \rangle + \Omega\omega_n \langle \phi_n \mathcal{I}_2(\phi_n, \Psi_p) \rangle + \Omega\omega_n \langle \phi_n \mathcal{I}_2(\Psi_p, \phi_n) \rangle = S_3, \quad (96)$$

where use of equation (37) was made.

7. CONCLUSIONS

A class of one-dimensional distributed-parameter systems with general quadratic and cubic geometric and quadratic inertia non-linearities has been the object of the analyses presented in this paper. Approximate results obtained for these systems, either for a primary resonance of a non-internally resonant mode or for a sub-harmonically excited two-to-one internal resonance, have been summarized. The results have been obtained with direct application of the method of multiple scales to the original governing equations, to the infinite-dimensional Galerkin-discretized system, and to low-order rectified Galerkin models. It is established, in a general fashion, the equivalence between the three procedures either in the case of no internal resonances or in the case of the two-to-one internal resonance. Here, the fact is emphasized that this equivalence can be shown to hold for more general resonance conditions, including multiple resonances. Interestingly, the rectified Galerkin discretization procedure, though producing low-dimensional models (the dimension being dependent on the internal resonance conditions), is capable of capturing and condensing correctly all of the modal contributions to the non-linear motions. Also, it is noted that this procedure is suitable for the analytical construction of non-linear normal modes of continuous systems with or without internal resonances.

From a computational standpoint, the direct and rectified Galerkin procedures seem to be more effective compared to the full-basis Galerkin discretization procedure. In fact, the latter requires an infinite-dimensional projection with subsequent solution of an infinite set of ordinary-differential equations at second order which is traded with solving, with less computational effort, some boundary-value problems in the direct and rectified Galerkin procedures. However, the full-basis Galerkin procedure furnishes a remarkably interesting spectral realization of the non-linear motions which can shed light on to the system behaviour.

Incidentally, it was shown that the peculiar effect of the quadratic non-linearities is to produce second-order contributions from all of the eigenmodes of the system to the non-linear motions. It was concluded that classical low-order Galerkin models may be inadequate to describe qualitatively the correct dynamics of the original system.

These results may be extended to more general non-linear non-self adjoint systems and general bidimensional distributed-parameter systems. In addition, it is suggested that these results be used as a baseline for convergence studies of low-dimensional discretized models of non-linear continuous systems.

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